Recitation 9

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Problems

Problem 1. With a little help of row reduction we get

$$\begin{vmatrix} 2 & -3 & 1 \\ -3 & 4 & 1 \\ 4 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 1 \\ -3 & 4 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1)^{3+2} \cdot \begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} = -1 \cdot (2+3) = -5$$

Problem 2. First we find eigenvalues. The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$, so $\lambda_1 = 2$ and $\lambda_2 = 3$ are the eigenvalues. Thus the diagonal matrix D from the decomposition $A = PDP^{-1}$ can be taken to be $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

If $\lambda = 2$, $A - 2I = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Thus when solving the system $(A - 2I)v = 0$ the variable x_2 is free, and $x_1 = x_2$. Thus eigenvectors corresponding to the eigenvalue $\lambda = 2$ are of the form $v_1 = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In particular we can take v_1 to be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We do the same thing for $\lambda = 3$, we get $A - 3I = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$. Thus eigenvectors corresponding to the eigenvalue $\lambda = 3$ are of the form $v_2 = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$. In particular we can take v_2 to be $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Thus matrix P can be taken to be $P = [v_1 v_2] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Thus $P^{-1} = \frac{1}{1\cdot 2 - 1\cdot 1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, and so

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Problem 3. The same stuff as before. First deal with $A - I$, and get

$$A - I = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Variables x_2, x_3 are free, and $x_1 = -2x_2 - x_3$, so the eigenvectors are of the form

$$v = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So we can take eigenvectors $v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Now do the same for $\lambda = 5$. You get

$$A - 5I = \begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & -4 & -4 \\ 0 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So x_3 is free, and $x_2 = -x_3$, $x_1 = 2x_2 + x_3 = -x_3$. So we can take the third eigenvector v_3 to be $v_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Thus we get

$$A = PDP^{-1} = \begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

Problem 4. The characteristic equation reads $\lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$ and so there is unique eigenvalue $\lambda = 4$. Then $A - 4I = \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. So all eigenvectors are of the form $v_1 = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since there is only one independent eigenvector, i.e. there is no basis consisting of eigenvectors of A, the matrix is not diagonalizable.

Problem 5. Put the vectors into matrix and row reduce

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since every row and column has a pivot, the two vectors form a basis of $\mathbb{R}^2$.

To find T in the **new** basis $\{v_1, v_2\}$ we find P, the matrix, columns of which is the coordinates of new basis in terms of the old basis, i.e. coordinates of v_1, v_2 in the standard (for this problem) basis. So $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Thus the matrix M of T in the new basis can be found using the matrix A of the transformation T in the old basis and the matrix P by the equation $M = P^{-1}AP$, i.e.

$$M = \frac{1}{1 \cdot 3 - 2 \cdot 2} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

Problem 6. We compute $T(-t^2 + 2t + 1) = t \cdot (-t^2 + 2t + 1)' = t(-2t + 2) = -2t^2 + 2t$.
To prove that T is a linear transformation, just check the definition.
$T(cp(t)) = t \cdot (cp(t))' = ctp(t)' = cT(p(t))$. Also
$T(p(t) + q(t)) = t \cdot (p(t) + q(t))' = tp(t)' + tq(t)' = T(p) + T(q)$. So T is a linear transformation.
To find the matrix, we need to see where the basis vectors go, and find their coordinates relative to the basis in the target space. We compute $T(1) = 0$, $T(t) = t$, $T(t^2) = 2t^2$. Thus the matrix is

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

To find null space we row reduce. Well, it's already row reduced. The first column has no pivots, so a basis for null space is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ which corresponds to the polynomial $p(t) \equiv 1 \in \mathbb{P}_2$. It's actually obvious that $T(a_0 + a_1 t + a_2 t^2) = a_1 t + 2a_2 t^2$, and so $T(p) = 0$ if and only if $a_1 = a_2 = 0$, so the null space consists of constant polynomials.
The rank of the matrix M above is 2, so the rank of T is 2. Rank doesn't depend on a basis, so it is enough to consider matrix of T in any basis you want.

Problem 7. You put the first basis (the basis in the domain) into the matrix P, and the basis of the codomain into the matrix Q, so

$$P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \ Q = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & -2 \end{bmatrix}$$

Then the matrix M of T in relative to the two bases will be $M = Q^{-1}AP$, i.e.

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 2 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Problem 8. If A has n distinct eigenvalues, then A can be diagonalized, and so $A = PDP^{-1}$ for a diagonal matrix D. Then $A^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D P^T$ where the last equality follows from the fact that D is diagonal, and so $D^T = D$, and also that transposition commutes with inverse, i.e. $(P^{-1})^T = (P^T)^{-1}$. But this means that in the basis given by the columns of $(P^T)^{-1}$ matrix A becomes the diagonal matrix D with n different entries along the diagonal. This exactly means that there is a basis on $\mathbb{R}^n$ where A^T acts by scaling this basis by different numbers. But this is just the definition of what is a basis of eigenvectors.

Problem 9. If $A = PBP^{-1}$ then

$$\det(A - \lambda I) = \det(PBP^{-1} - \lambda I) = \det(PBP^{-1} - \lambda PIP^{-1}) = \det P(B - \lambda I)P^{-1} = \det(B - \lambda I).$$

So the characteristic polynomials are the same.
If v is an eigenvector for A, then $Av = \lambda v$, and so $PBP^{-1}v = \lambda v$, and so $BP^{-1}v = \lambda P^{-1}v$. This exactly means that $P^{-1}v$ is an eigenvector of B with the same eigenvalue λ.